Why Are So Many Things in the Solar System Round?

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Several years ago a student asked why so many things in the solar system were round. He noted that many objects in the solar system, although not all, are round. The standard answer, which he knew, is that the mutual gravitational attraction of the molecules pulls them into the shape that gets them as close to each other as possible: a sphere. This argument works fine for fluid bodies such as the Sun or Jupiter, but it isn't so simple for a solid object—we have all seen rocks that are not round. There is still a gravitational attraction acting between the rock's molecules, but for small rocks that force does not overcome the strength of the bonds holding those molecules in their relative positions. Since the strength of the gravitational force grows with the size of the object, a large enough rock will have a strong enough gravitational attraction to force a deformation into a round shape. But how large is that? A simple model gives an answer to this question. There is also renewed interest in this topic as a result of the new definition of a planet approved by the International Astronomical Union, which says in part, "A 'planet' is a celestial body that … has sufficient mass for its self-gravity to overcome rigid body forces so that it assumes a hydrostatic equilibrium (nearly round) shape."[1] What size object is large enough to satisfy this criterion? Where does Pluto fall regarding this question?

The basic idea

Suppose I make a series of cubes of gelatin, each with a different size. Let's take the first cube to be 1 cm on a side, as shown in Fig. 1(a). Since gelatin is mostly water, we'll take the density to be 1 g/ml. Therefore, the mass of this cube is 1 g (10^{-3} kg), while the weight (= mass × gravitational acceleration) is 9.8×10^{-3} N. If you set this cube on a table, the weight is being supported by an area equal to the area of one face of the cube, 1 cm^2 (or 1×10^{-4} m^2). Pressure is defined as the force applied to a surface divided by the area on which the force is exerted, so in this case the pressure on the bottom of the cube is 9.8×10^{-3} N/1×10^{-4} m^2 = 98 N/m^2 = 98 Pa.

Next, let's see what happens if we double the size of the cube, as in Fig. 1(b). This time the volume is (2 cm)^3 = 8 cm^3. The resulting mass is now 8 g, and the resulting weight is 7.84×10^{-2} N. This weight is being supported by an area of 4 cm^2 (= 4×10^{-4} m^2), and so the pressure on the base is 196 Pa, twice as much as for the 1-cm cube. In general, since the weight scales as the cube of the scale factor and the area scales as the square, the pressure will grow linearly with the length of a side. A cube with an edge of 10 cm, Fig. 1(c), will have a pressure on its base 10 times that of our original cube.

But there is a limit to the pressure gelatin can withstand. If you push down on the cube of gelatin with your hand, it squeezes out the sides. The bonds holding it together as a solid are simply not strong enough to hold the gelatin together when the pressure is sufficiently great. At low pressures many materials will deform according to Hooke's law, so that they will spring back when the pressure is released. But beyond a limit (which varies with the material) the molecular bonds begin to break and the substance begins to flow. This limiting pressure that materials can withstand is called the *compressive strength*. When the cube of gelatin grows large enough that the pressure at the base exceeds the compressive strength (*CS*), the base begins to split and flow. Conclusion: There is a limit to the height that can be achieved with cubes of gelatin.

How tall can a mountain get?

The conditions are pretty tough under the bottom of a mountain. There is a lot of rock pressing down, and, just as with the gelatin, the base of a mountain must be strong enough to support the weight of the rock above it. Now mountains aren't cubical, so we'll use a more reasonable shape, but

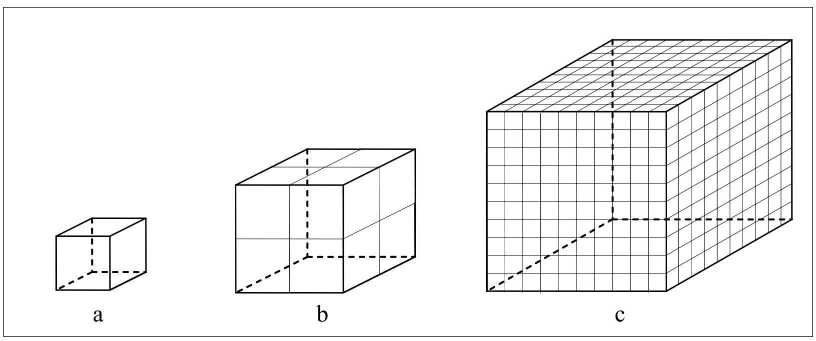

Fig. 1. Cubes of various sizes. In (a) we have a simple cube. The second cube (b) shows that a cube with twice the length of side will be composed of eight cubes like the original with four times the area on each face. The final cube (c) (not to scale) shows what you get when you use a scale factor of 10.

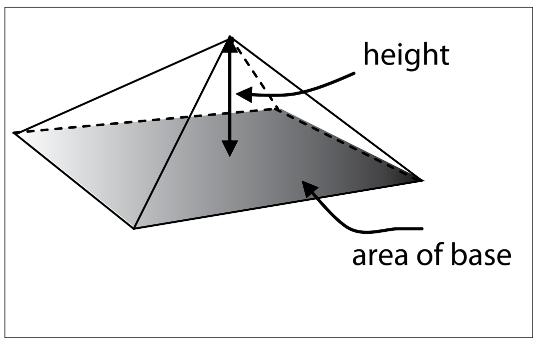

Fig. 2. The geometric definitions for a simple mountain.

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the analysis is the same. For convenience, let's say that mountains are pyramidal. The geometric parameters are defined in Fig. 2.

As before, the pressure P is calculated by the quotient of the weight W of the mountain and the area A of the base. The weight is mass M times g, and the mass is the density ρ times the volume V. The volume of a pyramid is $\frac{1}{3}A_{\text{base}}H$. Therefore, the pressure at the bottom is

$$P=\frac{W}{A_{\text{base}}}=\frac{Mg}{A_{\text{base}}}=\frac{\rho Vg}{A_{\text{base}}}=\frac{\rho\left(\frac{1}{3}A_{\text{base}}H\right)g}{A_{\text{base}}}.$$

Therefore, $P=\dfrac{\rho Hg}{3}$. (1)

This result differs from the case of the cube by the factor of 1/3. For whatever shape you desire for your mountain, there will be a similar constant multiplying the cube result.

There is a correction that can be made at this point regarding the pressure at the bottom of a pyramidal mountain. Since the mountain is not usually a single cohesive rock, the pressure at the bottom is not uniform across the whole base. At the edges the pressure is relatively low. A sand pile is perhaps a good example for modeling the pressure, and it has been found that the pressure at the base of a sand pile varies across the base. This variation differs from one sand pile to another based on the distribution of sizes of particles, but the maximum pressure is typically about twice what we would calculate from taking the total weight and dividing by the area of the base.[2] (An intuitive way of thinking about this is to say that since the pressure at the edges is low, the bulk of the weight of the sand pile is supported by about one-half the area of the base.) This modifies the equation for pressure by a factor of two:

$$P=\frac{2\rho Hg}{3}. \tag{2}$$

The mountain will be able to sustain itself so long as this pressure does not exceed the CS:

$$P=\frac{2\rho Hg}{3}\leq CS. \tag{3}$$

Solving for height, we find

$$H\leq\frac{3CS}{2\rho g}. \tag{4}$$

We see several relationships in this equation. For example, H is inversely proportional to the gravitational acceleration. Since the gravitational acceleration on Mars is about 0.38 times the value on Earth,[3] the maximum height of mountains on Mars (given the same density and CS) should be $\frac{1}{0.38}$ or 2.63 times the maximum height on Earth. There may not be any mountains at present that are at the limit, but let's check it out using the largest mountains each planet has to offer.

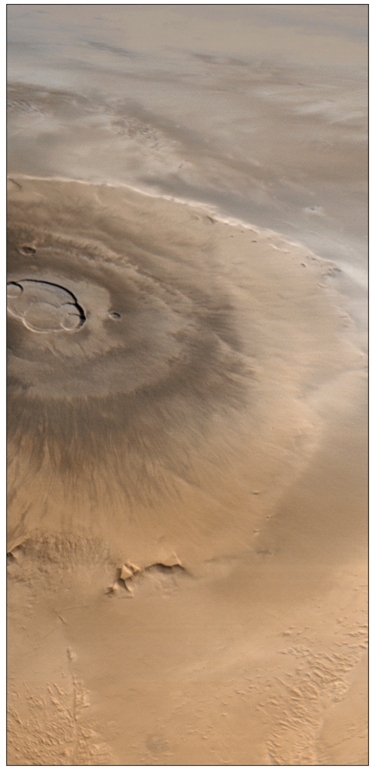


Fig. 3. Olympus Mons, the largest mountain on Mars (photo courtesy of NASA).

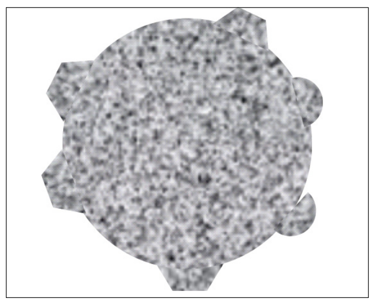
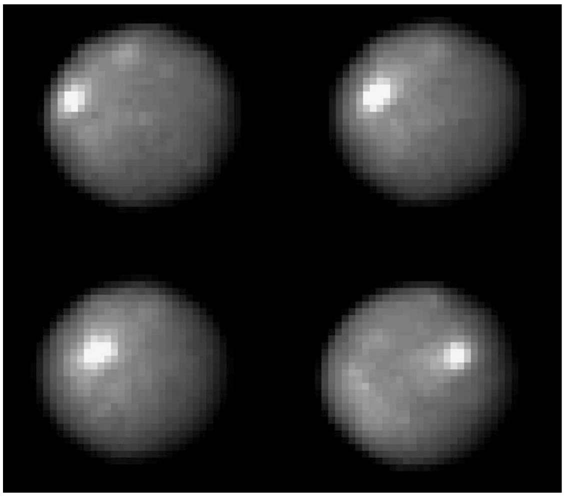


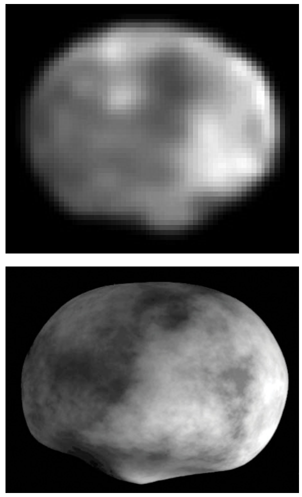
Fig. 4. This sketch shows a circle with various bumps attached that are 10% of the diameter in height. It is lumpy, but still roughly round.

The largest mountain on Mars is easy to spot—it is Olympus Mons, shown in Fig. 3, which reaches a height of 24 km above the surrounding plains (Fig. 4). For the Earth, you might expect us to use Mt. Everest, which reaches the highest altitude of all mountains. But there is a mountain that rises further above the surrounding land. It is harder to spot because much of it is underwater. Mauna Kea rises to a height of 10.2 km above the ocean floor. These two mountains give us a ratio of heights of 2.35. Not too bad.

This formula can also be used to predict the absolute maximum height of a mountain on the Earth (or Mars), although for this use there may be numerical factors we have ignored that will affect the precision of the prediction. For this we need to take the typical density and CS for the mate-



(a)



(b)

(c)

Fig. 5. Hubble images of Ceres (a) and Vesta (b), including a computer model of the shape of Vesta, and Eros (c), which was orbited by the *NEAR-Shoemaker* probe. (Images courtesy of NASA.)

rial of which mountains are made on a planet. On the Earth, a mountain like Mauna Kea is generally made of volcanic rocks. The density of such material is roughly 3000 kg/m^3 and its *CS* is roughly 200 MPa.[4] Using these values and $g = 9.8$ m/s^2, we find that the maximum height of a volcanic mountain on Earth $\cong$ 10 km, in agreement with the height of Mauna Kea.[5] For Mars, again assuming basalt, but this time using the smaller gravitational acceleration, we find that the maximum height $\cong$ 26 km, also in reasonable agreement. (As noted, the simplicity of the model doesn't really support this level of agreement, so this is a somewhat lucky result.)

For those interested, a sugar crystal has a density of 1587 kg/m^3 and a *CS* of 1000 MPa. (This hardness measurement was done on a single small crystal, and so may not be representative of a large accumulation of sucrose.) As a result, the maximum height of a mountain made of sucrose on the Earth would be roughly 96 km. That would certainly be a big rock candy mountain. (Thank you, Burl Ives.)

Why are planets round?

The limit to the height of a mountain connects nicely to the roundness of a planet. For the Earth, we've seen that the maximum height is about 10 km, compared to the radius of the planet of 6400 km. That means the biggest possible bump on the surface of the Earth is much smaller than the radius, leaving us with an Earth that looks very smooth. How smooth does a planet need to be to look round? An arbitrary definition is that the height of any bumps on the surface is less than 10% of the diameter of the object. While this sounds pretty big, you still get an object that is somewhat round, as shown in Fig. 4.

Our chosen condition for roundness is now that the maximum height of a mountain on a planetary body should be no more than 1/5 of the radius:

$$H_{\max} = \frac{3CS}{2\rho g} \leq \frac{R}{5}. \tag{5}$$

The height is now related to the radius, but the gravitational acceleration is too. From Newton's law of gravity,

$$g = G\frac{M_{\text{planet}}}{R^2}. \tag{6}$$

Plugging this into the equation above, we now have

$$H_{\max} = \frac{3(CS)R^2}{2\rho G M_{\text{planet}}} \leq \frac{R}{5}. \tag{7}$$

There is one more dependence on radius that we can include here and that is the mass of the planet, equal to the density times the volume. We will assume that the density of the planet is the same as the mountain.

$$M = \rho V = \rho\frac{4}{3}\pi R^3. \tag{8}$$

Substituting this into the above equation gives us

$$H_{\max} = \frac{3(CS)R^2}{2\rho G\left(\rho\frac{4}{3}\pi R^3\right)} = \frac{9CS}{8\pi G\rho^2 R} \leq \frac{R}{5}. \tag{9}$$

This can now be solved for the minimum radius:

$$R \geq \sqrt{\frac{45CS}{8\pi\rho^2 G}}. \tag{10}$$

For an object made of volcanic rock (CS = 200 MPa and ρ = 3000 kg/m^3), we find that the minimum radius is 770 km. How does this compare with real objects in the solar system? This size object falls near the large end of the range of asteroid sizes. We have the *Dawn* space probe on its way to orbit Vesta (arrival in August 2011) and then Ceres (arrival in February 2015), but we already have telescopic images of each of these objects. Ceres, the largest asteroid, has a radius of about 500 km and is reasonably round. The next asteroids in decreasing size, Pallas and Vesta, are not round by our definition and have a radius of about 250 km. Eros is a small asteroid, roughly 13 × 33 km. Figure 5 shows Hubble telescope images of Ceres and Vesta and an image from the *NEAR-Shoemaker*

Fig. 6. Pluto and Charon. Pluto appears round, although we can't be so sure about Charon. (Image courtesy of NASA.)

mission of the asteroid Eros. We can see that the break between spherical and irregular falls in the range of the larger asteroids, a few hundred km in radius.

What do we know about Pluto? From the orbit of Charon we can find the mass (and hence density). We believe the radius of Pluto to be roughly 1195 km, the mass to be 1.25 $\times 10^{22}$ kg, and its average density to be about 1750 kg/m^3. The composition of Pluto is believed to be about 30% water ice and 70% rock. The compressive strength of ice is in the neighborhood of 10 MPa. If we assume that a mountain will reach its limit when the ice begins to flow, we can use the *CS* of ice for the calculation of maximum height of a mountain on Pluto. Our formula gives 14.7 km, just over 1% of the radius, so Pluto fits the definition of a planet in regard to its shape. The best image we have for showing the shape of the limb of Pluto is from the Hubble telescope and is shown in Fig. 6. This image does make it appear that Pluto is spherical (although we can't say much about Charon from this image). When the New Horizons mission reaches Pluto, we will see both objects in much greater detail.

Conclusion

The weight of a mountain produces pressure at its base, and there is a limit to the pressure that materials can withstand due to the bonds that hold it together. When the mountain is sufficiently large, the pressure will exceed the compressive strength of the material of which it is made and the base of the mountain will flow, limiting the size of the mountain, just as was the case for the gelatin. This in turn shows what properties of an astronomical body determine whether it will necessarily be forced into a round shape by its self gravity.

This question opened up a discussion of basic material science applied to bodies in the solar system. It allowed my students to use what they had learned in physics in a relatively simple way that allowed them to make verifiable predictions.

Acknowledgment

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References

1. International Astronomical Union, www.iau.org, press release, 24 August 2006.
2. Liffman, Kurt et al., "Stress in Sandpiles," Second International Conference on CFD in Minerals and Process Industries, CSIRO, Melbourne, Australia (1999).
3. nssdc.gsfc.nasa.gov/planetary/. This is NASA's web page for physical data on the solar system.
4. The value used for the compressive strength of rocks came from a slideshow available at www.usouthal.edu/geography/allison/GY403/GY403_Lecture7_DynamicAnalysis.pdf. Another site with a table showing the compressive strength for a variety of common materials (but not rocks) is www.efunda.com.
5. Students may be concerned that since much of the mass of Mauna Kea is below water there will be a significant buoyant force reducing the pressure at the base. Buoyant forces are due to the pressure of the surrounding fluid pushing up on the bottom of an object being greater than the fluid's pressure pushing down on the top. Since there is no water pushing up on the bottom, the buoyant force does not enter into the calculation. The water above the flanks of Mauna Kea does still push down, adding to the pressure at the base, but this additional weight is small compared to the weight of the mountain itself.

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